

Snyder-type spaces, twisted Poincaré algebra and addition of momenta

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Abstract

We discuss a generalisation of the Snyder model that includes all the possible deformations of the Heisenberg algebra compatible with Lorentz invariance, in terms of realisations of the noncommutative geometry. The corresponding deformed addition of momenta, the twist and the R -matrix are calculated to first order in the deformation parameters for all models. In the particular case of the Snyder realisation, the exact formula for the twist is obtained.

1 Introduction

In his seminal paper [1], Snyder observed that, assuming a noncommutative structure of spacetime, and hence a deformation of the Heisenberg algebra, it is possible to define a discrete spacetime without breaking the Lorentz invariance. In this way the short-distance behavior of quantum field theory can be improved, possibly avoiding ultraviolet divergences.

More recently, noncommutative geometry has become an important field of research [2]. New models have been introduced, as for example the Moyal plane [3] and κ -Minkowski geometry [4], and the formalism of Hopf algebras has been applied to their study [5]. However, contrary to Snyder's, these models either break or deform the action of the Lorentz group on spacetime.

It is therefore interesting to investigate the Snyder model from the point of view of noncommutative geometry. The Hopf algebra associated with the Snyder model has been studied in a series of papers [6, 7, 8], where the model has been generalised and the star product, coproduct and antipodes have been calculated using the method of realisations. A different approach was used in [9], where the Snyder model was considered in a geometrical perspective as a coset in momentum space, and results equivalent to those of refs. [6, 7] were obtained. More recently, in [10] a further generalisation was introduced and the construction of QFT on Snyder spacetime was undertaken.

However, some basic properties of the Hopf algebra formalism for Snyder spaces have not yet been investigated: for example the twist and the related R -matrix have not been explicitly calculated, although they have proven to be very useful tools, especially in the construction of a QFT. In particular, the knowledge of the R -matrix is useful for the definition of a twisted statistics in QFT. Actually, some difficulties arise because the coproduct in Snyder spaces is non-coassociative, so that the twist will not satisfy the cocycle condition for the Hopf algebra.

From a different point of view, phenomenological aspects of the Snyder model have been investigated in classical and quantum physics, especially in the nonrelativistic 3D limit [11, 12, 13]. The most interesting results in this context are the clarification of its lattice-like properties, leading to deformed uncertainty relations, and the study of the corrections induced on the energy spectrum of some simple physical systems.

In this paper, we extend previous investigations on the noncommutative geometry of the generalised Snyder models, by calculating the twist and the R -matrix to first order in the deformation parameter in the general case. We also obtain the exact expression of the twist for the so-called Snyder realisation, introduced in the original paper [1].

We note that our results could be rephrased using the formalism of Hopf algebroids [14], which is for some aspects more suitable for the description of the Snyder models than the usual one based on Hopf algebras, since it deals with the full phase space; however we leave this subject to future investigations.

2 Snyder space and its generalisation

Generalised Snyder spaces are a deformation of ordinary phase space, generated by noncommutative coordinates \bar{x}_μ and momenta p_μ that span a deformed Heisenberg

algebra $\tilde{\mathcal{H}}(\bar{x}, p)$,

$$[\bar{x}_\mu, \bar{x}_\nu] = i\beta M_{\mu\nu} \psi(\beta p^2), \quad [p_\mu, p_\nu] = 0, \quad [p_\mu, \bar{x}_\nu] = -i\varphi_{\mu\nu}(\beta p^2), \quad (1)$$

together with Lorentz generators $M_{\mu\nu}$ that satisfy the standard relations

$$\begin{aligned} [M_{\mu\nu}, M_{\rho\sigma}] &= i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\nu\sigma}M_{\mu\rho}), \\ [M_{\mu\nu}, p_\lambda] &= i(\eta_{\mu\lambda}p_\nu - \eta_{\lambda\nu}p_\mu), \quad [M_{\mu\nu}, \bar{x}_\lambda] = i(\eta_{\mu\lambda}\bar{x}_\nu - \eta_{\nu\lambda}\bar{x}_\mu), \end{aligned} \quad (2)$$

where the functions $\psi(\beta p^2)$ and $\varphi_{\mu\nu}(\beta p^2)$ are constrained so that the Jacobi identities hold, β is a constant of the order of $1/M_{Pl}^2$, and $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. The commutation relations (1)-(2) generalize those originally investigated in [1], that are recovered for $\psi = \text{const}$.

We recall that in its undeformed version, the Heisenberg algebra $\mathcal{H}(x, p)$ is generated by commutative coordinates x_μ and momenta p_μ , satisfying

$$[x_\mu, x_\nu] = [p_\mu, p_\nu] = 0, \quad [p_\mu, x_\nu] = -i\eta_{\mu\nu}. \quad (3)$$

The action of x_μ and p_μ on functions $f(x)$ belonging to the enveloping algebra \mathcal{A} generated by the x_μ is defined as

$$x_\mu \triangleright f(x) = x_\mu f(x), \quad p_\mu \triangleright f(x) = -i \frac{\partial f(x)}{\partial x^\mu}. \quad (4)$$

The noncommutative coordinates \bar{x}_μ and the Lorentz generators $M_{\mu\nu}$ in (1)-(2) can be expressed in terms of commutative coordinates x_μ and momenta p_μ as [6, 7]

$$\bar{x}_\mu = x_\mu \varphi_1(\beta p^2) + \beta x \cdot p p_\mu \varphi_2(\beta p^2) + \beta p_\mu \chi(\beta p^2), \quad (5)$$

$$M_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu. \quad (6)$$

Notice that the function χ does not appear in the defining relations (1)-(2), but takes into account ambiguities arising from operator ordering of x_μ and p_μ in equation (5).

In terms of the realisation (5), the functions $\varphi_{\mu\nu}$ in (1) read

$$\varphi_{\mu\nu} = \eta_{\mu\nu} \varphi_1 + \beta p_\mu p_\nu \varphi_2, \quad (7)$$

while the Jacobi identities are satisfied if

$$\psi = -2\varphi_1 \varphi_1' + \varphi_1 \varphi_2 - 2\beta p^2 \varphi_1' \varphi_2, \quad (8)$$

where the prime denotes a derivative with respect to βp^2 . In particular, the function ψ does not depend on the function χ .

From (8) it follows that the coordinates \bar{x}_μ are commutative for $\varphi_2 = \frac{2\varphi_1' \varphi_1}{\varphi_1 - 2\beta p^2 \varphi_1'}$, and correspond to Snyder space for $\varphi_2 = \frac{1+2\varphi_1' \varphi_1}{\varphi_1 - 2\beta p^2 \varphi_1'}$. In particular, the Snyder realisation [1] is recovered for $\varphi_1 = \varphi_2 = 1$, and the Maggiore [15] realisation for $\varphi_1 = \sqrt{1 - \beta p^2}$, $\varphi_2 = 0$ [6, 7]. There is also another interesting exact realisation of Snyder space for $\psi = s = \text{const}$, given by

$$\bar{x}_\mu = x_\mu + \frac{\beta s}{4} K_\mu, \quad (9)$$

where $K_\mu = x_\mu p^2 - 2x \cdot p p_\mu$ are the generators of conformal transformations in momentum space, with $[K_\mu, K_\nu] = 0$. The algebra (1) unifies commutative space, $\psi = 0$, and Snyder space, $\psi = 1$. Since the Lorentz transformations are not deformed, the Casimir operator of the algebra (1)-(2) is $C = p^2$.

The Hopf algebra associated with these spaces can be investigated using the formalism introduced in refs. [16, 17, 18] and shortly reviewed in [20], to which we refer for more details. It turns out that the generalized addition of momenta k_μ and q_μ is given by [7, 8, 19]

$$k_\mu \oplus q_\mu = \mathcal{D}_\mu(k, q), \quad \text{with} \quad \mathcal{D}_\mu(k, 0) = k_\mu, \quad \mathcal{D}_\mu(0, q) = q_\mu, \quad (10)$$

where $k, q \in M_{1,3}$. The function $\mathcal{D}_\mu(k, q)$ can be calculated in terms of $\varphi_{\mu\nu}$ as

$$\mathcal{D}_\mu(k, q) = \mathcal{P}_\mu(\mathcal{K}^{-1}(k), q), \quad (11)$$

where we have introduced a function $\mathcal{P}_\mu(\lambda k, q)$ that satisfies the differential equation

$$\frac{d\mathcal{P}_\mu(\lambda k, q)}{d\lambda} = k_\alpha \varphi_\mu^\alpha(\mathcal{P}(\lambda k, q)), \quad (12)$$

with

$$\mathcal{P}_\mu(k, 0) = \mathcal{K}_\mu(k), \quad \mathcal{P}_\mu(0, q) = q_\mu, \quad (13)$$

$\mathcal{K}_\mu^{-1}(k)$ being the inverse map of $\mathcal{K}_\mu(k)$, i.e. $\mathcal{K}_\mu^{-1}(\mathcal{K}(k)) = k_\mu$, and λ a real parameter. From (12) and (7) it follows that $\mathcal{P}_\mu(k, q)$ and hence $\mathcal{D}_\mu(k, q)$ do not depend on the function χ in (5).

It can be shown that [16, 17, 18]

$$e^{ik \cdot \tilde{x}} \triangleright e^{iq \cdot x} = e^{i\mathcal{P}(k, q) \cdot x + iQ(k, q)}, \quad (14)$$

where $Q(k, q)$ satisfies the differential equation

$$\frac{dQ(\lambda k, q)}{d\lambda} = k_\alpha \chi^\alpha(\mathcal{P}(\lambda k, q)), \quad (15)$$

with $Q(0, q) = 0$ and $\chi^\alpha \equiv p^\alpha \chi(\beta p^2)$.

Calculating the star product of two plane waves one then obtains

$$e^{ik \cdot x} \star e^{iq \cdot x} = e^{i\mathcal{D}(k, q) \cdot x + i\mathcal{G}(k, q)}, \quad (16)$$

with

$$\mathcal{G}(k, q) = Q(\mathcal{K}^{-1}(k), q) - Q(\mathcal{K}^{-1}(k), 0). \quad (17)$$

Note that \mathcal{G} vanishes if $\chi(k) = 0$.

The algebra \mathcal{A} can be extended to the algebra \mathcal{U} generated by the x_μ and the p_μ , symbolically indicated as $\mathcal{U} = \mathcal{A}\mathcal{T}$, where \mathcal{T} is the algebra generated by the p_μ [14]. The coproduct for the momenta Δp_μ , is obtained from $\mathcal{D}_\mu(k, q)$ as

$$\Delta p_\mu = \mathcal{D}_\mu(p \otimes 1, 1 \otimes p). \quad (18)$$

Notice that the previous definitions imply that the addition of momenta and the coproduct do not depend on $\chi(\beta p^2)$.

From the coproduct one can then define the twist \mathcal{F} , such that $\Delta h = \mathcal{F} \Delta_0 h \mathcal{F}^{-1}$ for any $h \in \mathcal{U}$, as [14, 21, 22]

$$\mathcal{F}^{-1} =: \exp[i(1 \otimes x_\alpha)(\Delta - \Delta_0)p_\alpha + \mathcal{G}(p \otimes 1, 1 \otimes p)] : , \quad (19)$$

where $\Delta_0 p_\mu = p_\mu \otimes 1 + 1 \otimes p_\mu$, and $: :$ denotes normal ordering in which the coordinates x_α stand on the left of the momenta p_α .

The star product $f \star g$ can be defined as

$$(f \star g)(x) = m(\mathcal{F}^{-1}(\triangleright \otimes \triangleright)(f \otimes g)), \quad f, g \in \mathcal{A}, \quad (20)$$

with $m : \mathcal{A} \otimes \mathcal{H} \rightarrow \mathcal{H}$ the multiplication map of \mathcal{A} .

The relation (5) between \bar{x}_μ and x_μ can also be written in terms of the twist as

$$\bar{x}_\mu = m(\mathcal{F}^{-1}(\triangleright \otimes 1)(x_\mu \otimes 1)) = x_\alpha \varphi_\mu^\alpha(p) + \beta p_\mu \chi(p). \quad (21)$$

It follows for consistency

$$\Delta p_\mu = \mathcal{F}(\Delta_0 p_\mu) \mathcal{F}^{-1}, \quad \Delta_0 p_\mu = p_\mu \otimes 1 + 1 \otimes p_\mu \quad (22)$$

in accordance with (18).

The coproducts of momenta are found for special cases in [7]: for the Snyder realisation

$$\Delta p_\mu = \frac{1}{1 - \beta p_\alpha \otimes p^\alpha} \left(p_\mu \otimes 1 - \frac{\beta}{1 + \sqrt{1 + \beta p^2}} p_\mu p_\alpha \otimes p^\alpha + \sqrt{1 + \beta p^2} \otimes p_\mu \right), \quad (23)$$

while for the Maggiore realisation

$$\Delta p_\mu = p_\mu \otimes \sqrt{1 - \beta p^2} - \frac{\beta}{1 + \sqrt{1 - \beta p^2}} p_\mu p_\alpha \otimes p^\alpha + 1 \otimes p_\mu. \quad (24)$$

The coproducts of the Lorentz generators are instead

$$\Delta M_{\mu\nu} = \mathcal{F}(\Delta_0 M_{\mu\nu}) \mathcal{F}^{-1}, \quad \Delta_0 M_{\mu\nu} = M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu}. \quad (25)$$

Because of the commutation relations (2), the coproduct of $M_{\mu\nu}$ will be trivial, i.e. $\Delta M_{\mu\nu} = \Delta_0 M_{\mu\nu}$ [7].

We recall that also the antipodes for Snyder space are trivial [7],

$$S(p_\mu) = -p_\mu, \quad S(M_{\mu\nu}) = -M_{\mu\nu}, \quad (26)$$

3 First order expansion

The study of the general form of the deformed Heisenberg algebra (1) is difficult, however one can study it perturbatively, by expanding the realisation (5) of the noncommutative coordinates in powers of β , namely,

$$\bar{x}_\mu = x_\mu + \beta(s_1 x_\mu p^2 + s_2 x \cdot p p_\mu + c p_\mu) + o(\beta^2), \quad (27)$$

with parameters s_1, s_2, c . Hence, the commutation relations do not depend on the parameter c and to first order are given by

$$[\bar{x}_\mu, \bar{x}_\nu] = i\beta s M_{\mu\nu} + o(\beta^2), \quad [p_\mu, \bar{x}_\nu] = -i[\eta_{\mu\nu}(1 + \beta s_1 p^2) + \beta s_2 p_\mu p_\nu] + o(\beta^2), \quad (28)$$

where $s = s_2 - 2s_1$.

The models of ref. [6, 7] are recovered for $s_2 = 1 + 2s_1$. Moreover, for $s_1 = 0, s_2 = 1$, eqs. (27)-(28) reproduce the exact Snyder realisation, while for $s_1 = -\frac{1}{2}, s_2 = 0$ they give the first-order expansion of the Maggiore realisation. For $s_2 = 2s_1$, spacetime is commutative to first order in β , while for $s_1 = -s/4, s_2 = s/2, c = 0$ one gets the exact realisation (9).

The first order expression for the function $\mathcal{P}_\mu(k, q)$ is given by

$$\begin{aligned} \mathcal{P}_\mu(k, q) &= q_\mu + \int_0^1 d\lambda \left\{ k_\mu + \beta \left[s_1 k_\mu (\lambda k + q)^2 + s_2 (\lambda k^2 + k \cdot q) (\lambda k_\mu + q_\mu) \right] \right\} + o(\beta^2) \\ &= k_\mu + q_\mu + \beta \left[\left(s_1 q^2 + \left(s_1 + \frac{s_2}{2} \right) k \cdot q + \frac{s_1 + s_2}{3} k^2 \right) k_\mu + s_2 \left(k \cdot q + \frac{k^2}{2} \right) q_\mu \right] \\ &\quad + o(\beta^2), \end{aligned} \quad (29)$$

from where it follows that

$$\mathcal{K}_\mu^{-1}(k) = k_\mu - \frac{\beta}{3}(s_1 + s_2)k^2 k_\mu + o(\beta^2). \quad (30)$$

These results allow us to write down the generalized addition law of the momenta k_μ and q_μ at first order

$$(k \oplus q)_\mu = \mathcal{D}_\mu(k, q) = k_\mu + q_\mu + \beta \left[s_2 k \cdot q q_\mu + s_1 q^2 k_\mu + \left(s_1 + \frac{s_2}{2} \right) k \cdot q k_\mu + \frac{s_2}{2} k^2 q_\mu \right] + o(\beta^2). \quad (31)$$

In particular, for the "conformal" case (9) with parameters $s_1 = -s/4, s_2 = s/2$,

$$(k \oplus q)_\mu = k_\mu + q_\mu + \frac{\beta s}{4} [2 k \cdot q q_\mu - q^2 k_\mu + k^2 q_\mu] + o(\beta^2). \quad (32)$$

It is also interesting to remark that for $s_2 = 2s_1 \neq 0, s = 0$, although spacetime is commutative up to the first order in β , the addition of momenta is deformed,

$$(k \oplus q)_\mu \neq k_\mu + q_\mu. \quad (33)$$

The Lorentz transformations of momenta are not deformed, and denoting them by $\Lambda(\xi, p)$, with ξ the rapidity parameter, the law of addition of momenta implies that

$$\Lambda(\xi, k \oplus q) = \Lambda(\xi_1, k) \oplus \Lambda(\xi_2, q) \quad (34)$$

is satisfied for $\xi_1 = \xi_2 = \xi$. Hence there are no backreaction factors in the sense of ref. [23, 24]. This means that in composite systems the boosted momenta of the single particles are independent of the momenta of the other particles in the system.

The coproduct to first order can be read from (31) and is given by

$$\Delta p_\mu = \Delta_0 p_\mu + \beta \left[s_1 p_\mu \otimes p^2 + s_2 p_\alpha \otimes p^\alpha p_\mu + \left(s_1 + \frac{s_2}{2} \right) p_\mu p_\alpha \otimes p^\alpha + \frac{s_2}{2} p^2 \otimes p_\mu \right] + o(\beta^2). \quad (35)$$

The corresponding twist operator \mathcal{F}^{-1} is

$$\mathcal{F}^{-1} = 1 \otimes 1 + i(1 \otimes x_\alpha)(\Delta - \Delta_0)p^\alpha + ic\beta p_\alpha \otimes p^\alpha + o(\beta^2). \quad (36)$$

From this one can calculate the coproduct $\Delta M_{\mu\nu}$, and the antipodes $S(p_\mu)$ and $S(M_{\mu\nu})$.

In general, the exact twist will not satisfy the cocycle condition, the star product will be non-associative and the coproducts Δp_μ , $\Delta M_{\mu\nu}$ will be non-coassociative [7], so that the corresponding structure is a quasi-Hopf algebra. An exception is given by the commutative case $s_2 = 2s_1$, when the star product is associative and the corresponding coproduct Δp_μ is cocommutative and coassociative

Using the twist (36) to calculate the coproduct of p_μ as in (22), one gets again (35), the same result as when using the function \mathcal{D} , while using (25) to calculate the coproduct of $M_{\mu\nu}$ gives $\Delta M_{\mu\nu} = \Delta_0 M_{\mu\nu} + o(\beta^2)$.

In the special case $s_2 = 2s_1$, $s = 0$, which corresponds to commutative space, it is easily seen from (35) that

$$\tilde{\Delta} p_\mu \equiv \tau_0 \Delta p_\mu \tau_0 = \Delta p_\mu, \quad (37)$$

i.e. the coproduct is left-right symmetric, with the flip operator τ_0 defined in the usual way as

$$\tau_0(A \otimes B) = B \otimes A. \quad (38)$$

The coproduct is also cocommutative and the corresponding star product is commutative, but not local.

The flip operator, $\tau = \mathcal{F} \tau_0 \mathcal{F}^{-1}$, is relevant in the discussion of the twisted statistics of particles in quantum field theory on noncommutative spaces [21, 22]. Another important operator in this context is the R -matrix, which satisfies the relation $R \Delta_0 p_\mu R^{-1} = \tilde{\Delta} p_\mu$. Defining $\tilde{\mathcal{F}} = \tau_0 \mathcal{F} \tau_0$, it can be written as

$$R = \tilde{\mathcal{F}} \mathcal{F}^{-1} = 1 \otimes 1 + R_{cl} + o(\beta^2), \quad (39)$$

where the classical R -matrix R_{cl} is

$$R_{cl} = (x_\alpha \otimes 1)(\tilde{\Delta} - \Delta_0)p^\alpha - (1 \otimes x_\alpha)(\Delta - \Delta_0)p^\alpha, \quad (40)$$

where Δp_μ is given in (35). For commutative spaces, for which (37) holds, R_{cl} is given by

$$R_{cl} = (x_\alpha \otimes 1 - 1 \otimes x_\alpha)(\Delta - \Delta_0)p^\alpha \in \mathcal{I}_0. \quad (41)$$

where \mathcal{I}_0 is the right ideal of \mathcal{U} with the property $m(\mathcal{I}_0 \triangleright (f \otimes g)) = 0$.

4 Twist for the Snyder realisation

In this section, we construct the exact twist operator for the Snyder space using the perturbative approach introduced in [14], by expanding (22) in powers of β . We first

consider the special case of the Snyder realisation $\varphi_1 = \varphi_2 = 1, \chi = 0$, for which

$$\bar{x}_\mu = x_\mu + \beta x \cdot p p_\mu. \quad (42)$$

The coproduct of the momenta is given by (23). We expand it with respect to the deformation parameter β as $\Delta p_\mu = \sum_{k=0}^{\infty} \Delta_k p_\mu$, with $\Delta_k p_\mu \propto \beta^k$

$$\begin{aligned} \Delta p_\mu &= p_\mu \otimes 1 + 1 \otimes p_\mu + \beta \left(\frac{1}{2} p_\mu p_\alpha \otimes p^\alpha + p_\alpha \otimes p^\alpha p_\mu + \frac{1}{2} p^2 \otimes p_\mu \right) \\ &+ \beta^2 \left(\frac{1}{2} p_\mu p_\alpha p_\beta \otimes p^\alpha p^\beta + p_\alpha p_\beta \otimes p^\alpha p^\beta p_\mu + \frac{1}{8} p_\mu p_\alpha p^2 \otimes p^\alpha - \frac{1}{8} p^4 \otimes p_\mu \right. \\ &+ \left. \frac{1}{2} p_\alpha p^2 \otimes p^\alpha p_\mu \right) + \beta^3 \left(\frac{1}{2} p_\mu p_\alpha p_\beta p_\gamma \otimes p^\alpha p^\beta p^\gamma + p_\alpha p_\beta p_\gamma \otimes p^\alpha p^\beta p^\gamma p_\mu \right. \\ &- \left. \frac{1}{16} p_\mu p_\alpha p^4 \otimes p^\alpha + \frac{1}{8} p_\mu p_\alpha p_\beta p^2 \otimes p^\alpha p^\beta + \frac{1}{16} p^6 \otimes p_\mu - \frac{1}{8} p_\alpha p^4 \otimes p^\alpha p_\mu \right. \\ &+ \left. \frac{1}{2} p_\alpha p_\beta p^2 \otimes p^\alpha p^\beta p_\mu \right) + o(\beta^4) \end{aligned} \quad (43)$$

and we look for the twist operator in the form

$$\mathcal{F} = e^{f_1 + f_2 + f_3 + \dots}, \quad (44)$$

where $f_k \propto \beta^k$. From (22) we obtain the equations satisfied by the f_k order by order,

$$[f_1, \Delta_0 p_\mu] = \Delta_1 p_\mu, \quad (45)$$

$$[f_2, \Delta_0 p_\mu] = \Delta_2 p_\mu - \frac{1}{2} [f_1, [f_1, \Delta_0 p_\mu]], \quad (46)$$

$$\begin{aligned} [f_3, \Delta_0 p_\mu] &= \Delta_3 p_\mu - \frac{1}{2} ([f_1, [f_2, \Delta_0 p_\mu]] + [f_2, [f_1, \Delta_0 p_\mu]]) \\ &- \frac{1}{3!} [f_1, [f_1, [f_1, \Delta_0 p_\mu]]], \end{aligned} \quad (47)$$

and so on. To calculate f_1 we write down the ansatz

$$f_1 = \beta (\alpha_1 p^2 \otimes x \cdot p + \alpha_2 p_\alpha p_\beta \otimes x^\alpha p^\beta + \alpha_3 p_\alpha \otimes x \cdot p p^\alpha + \alpha_4 p_\alpha \otimes x^\alpha p^2)$$

and insert it into (45) to determine the unknown coefficients α_i . The resulting expression for f_1 is

$$f_1 = -i\beta \left(\frac{1}{2} p^2 \otimes x \cdot p + \frac{1}{2} p_\alpha p_\beta \otimes x^\alpha p^\beta + p_\alpha \otimes x \cdot p p^\alpha \right). \quad (48)$$

Inserting this and the ansatz

$$\begin{aligned} f_2 &= \beta^2 (\alpha_1 p^4 \otimes x \cdot p + \alpha_2 p_\alpha p_\beta p^2 \otimes x^\alpha p^\beta + \alpha_3 p_\alpha p^2 \otimes x \cdot p p^\alpha \\ &+ \alpha_4 p_\alpha p^2 \otimes x^\alpha p^2 + \alpha_5 p_\alpha p_\beta p_\gamma \otimes x^\alpha p^\beta p^\gamma), \end{aligned}$$

into (46), we find

$$f_2 = i \frac{\beta^2}{2} \left(\frac{1}{2} p^4 \otimes x \cdot p + \frac{1}{2} p_\alpha p_\beta p^2 \otimes x^\alpha p^\beta + p_\alpha p^2 \otimes x \cdot p p^\alpha \right). \quad (49)$$

An analogous procedure to third order gives

$$f_3 = -i \frac{\beta^3}{3} \left(\frac{1}{2} p^6 \otimes x \cdot p + \frac{1}{2} p_\alpha p_\beta p^4 \otimes x^\alpha p^\beta + p_\alpha p^4 \otimes x \cdot p p^\alpha \right). \quad (50)$$

Inductively, we get a closed form for the twist

$$\mathcal{F} = \exp \left\{ -i \left(\frac{1}{2} p^2 \otimes x \cdot p + \frac{1}{2} p_\alpha p_\beta \otimes x^\alpha p^\beta + p_\alpha \otimes x \cdot p p^\alpha \right) \left(\frac{\log(1 + \beta p^2)}{p^2} \otimes 1 \right) \right\}. \quad (51)$$

One can check that (51) gives the correct twist for the Snyder space by calculating

$$m(\mathcal{F}^{-1}(\triangleright \otimes 1)(x_\mu \otimes 1)) = x_\mu + \beta x \cdot p p_\mu. \quad (52)$$

An independent verification is to start from (19). We get

$$\begin{aligned} \mathcal{F}^{-1} = : \exp & \left[\frac{i}{1 - \beta p_\alpha \otimes p^\alpha} \left(\frac{\beta \sqrt{1 + \beta p^2}}{1 + \sqrt{1 + \beta p^2}} p^\mu p^\nu \otimes x_\mu p_\nu + \left(\sqrt{1 + \beta p^2} - 1 \right) \otimes x \cdot p \right. \right. \\ & \left. \left. + \beta p_\alpha \otimes x \cdot p p^\alpha \right) \right] : , \end{aligned} \quad (53)$$

which expanded up to second order gives

$$\begin{aligned} \mathcal{F}^{-1} &= 1 \otimes 1 + i\beta \left(\frac{1}{2} p^\alpha p^\beta \otimes x_\alpha p_\beta + \frac{1}{2} p^2 \otimes x \cdot p + p_\alpha \otimes x \cdot p p^\alpha \right) \\ &- \frac{i\beta^2}{2} \left(\frac{1}{4} p^4 \otimes x \cdot p - \frac{1}{4} p_\alpha p_\beta p^2 \otimes x^\alpha p^\beta - p_\alpha p^2 \otimes x \cdot p p^\alpha - p_\alpha p_\beta p_\gamma \otimes x^\alpha p^\beta p^\gamma \right. \\ &- \left. 2p_\alpha p_\beta \otimes x \cdot p p^\alpha p^\beta \right) - \frac{\beta^2}{2} \left(\frac{1}{4} p^4 \otimes x^\alpha x \cdot p p_\alpha + \frac{1}{2} p_\alpha p_\beta p^2 \otimes x^\alpha x \cdot p p^\beta \right. \\ &+ \left. p_\alpha p^2 \otimes x_\beta x \cdot p p^\beta p^\alpha + \frac{1}{4} p_\alpha p_\beta p_\gamma p_\delta \otimes x^\alpha x^\beta p^\gamma p^\delta + p_\alpha p_\beta p_\gamma \otimes x^\alpha x \cdot p p^\beta p^\gamma \right. \\ &+ \left. p_\alpha p_\beta \otimes x_\gamma x \cdot p p^\gamma p^\alpha p^\beta \right) + o(\beta^3). \end{aligned} \quad (54)$$

The expression in eq. (54) agrees exactly with what one would get from (48) and (49) using the fact that $\mathcal{F}^{-1} = 1 \otimes 1 - f_1 - f_2 + \frac{1}{2} f_1^2 + o(\beta^3)$.

Using the exact twist (51) to calculate the coproduct of $M_{\mu\nu}$ we can verify that the coproduct of the Lorentz generators is undeformed to all orders i.e.,

$$\Delta M_{\mu\nu} = \Delta_0 M_{\mu\nu}. \quad (55)$$

Note that exact twist corresponding to the Snyder realization can be written in terms of the dilatation $D = x \cdot p$ and of p^2 , in a form which slightly differs from eq. (51) but is equivalent to it.

5 Twist for the Maggiore realisation

The same procedure can be performed for the Maggiore realisation (24). The coproduct, when expanded up to the third order, takes the following form

$$\begin{aligned}\Delta p_\mu &= p_\mu \otimes 1 + 1 \otimes p_\mu - \frac{\beta}{2} (p_\mu p_\alpha \otimes p^\alpha + p_\mu \otimes p^2) \\ &\quad - \frac{\beta^2}{8} (p_\mu \otimes p^4 + p_\mu p_\alpha p^2 \otimes p^\alpha) - \frac{\beta^3}{16} (p_\mu \otimes p^6 + p_\mu p_\alpha p^4 \otimes p^\alpha) + o(\beta^4)\end{aligned}\quad (56)$$

Using the same procedure as in the previous section, we find

$$\begin{aligned}f_1 &= \frac{i\beta}{2} (p_\alpha \otimes x^\alpha p^2 + p_\alpha p_\beta \otimes x^\alpha p^\beta), \\ f_2 &= \frac{i\beta^2}{8} (p_\alpha \otimes x^\alpha p^4 + p_\alpha p^2 \otimes x^\alpha p^2 + 2p_\alpha p_\beta p_\gamma \otimes x^\alpha p^\beta p^\gamma \\ &\quad + 2p_\alpha p_\beta \otimes x^\alpha p^\beta p^2 + 2p_\alpha p_\beta p^2 \otimes x^\alpha p^\beta), \\ f_3 &= \frac{i\beta^3}{8} \left(\frac{1}{2} p_\alpha \otimes x^\alpha p^6 + \frac{4}{3} p_\alpha p_\beta p^4 \otimes x^\alpha p^\beta + \frac{3}{2} p_\alpha p_\beta \otimes x^\alpha p^\beta p^4 \right. \\ &\quad \left. + \frac{7}{12} p_\alpha p^4 \otimes x^\alpha p^2 + \frac{5}{12} p_\alpha p^2 \otimes x^\alpha p^4 + \frac{7}{3} p_\alpha p_\beta p_\gamma \otimes x^\alpha p^\beta p^\gamma p^2 \right. \\ &\quad \left. + \frac{5}{3} p_\alpha p_\beta p^2 \otimes x^\alpha p^\beta p^2 + \frac{4}{3} p_\alpha p_\beta p_\gamma p_\delta \otimes x^\alpha p^\beta p^\gamma p^\delta + 2p_\alpha p_\beta p_\gamma p^2 \otimes x^\alpha p^\beta p^\gamma \right).\end{aligned}\quad (57)$$

In this case, we were not able to find a closed form for the twist. However, the perturbative result, when used to calculate the coproduct of $M_{\mu\nu}$, gives again the primitive coproduct.

We also recall that, because of the non-coassociativity of the coproduct, the twist for the Snyder space will not satisfy the cocycle condition for the Hopf algebra, as can be checked directly order by order.

6 Conclusions

In this paper we have investigated the most general realisations of the Snyder model compatible with undeformed Lorentz invariance, and have calculated the twist and the R-matrix for the generic case, at leading order in the deformation parameters. In particular, in the specific case of the Snyder realisation we were able to obtain the exact expression for the twist.

Our results can be rephrased using the formalism of Hopf algebroids [14], that is for some aspects more suitable for the description of the Snyder models than the usual one based on Hopf algebras, since it deals with the full phase space. We leave this subject to future investigations.

The results obtained in this paper may be important for the construction of a complete QFT on Snyder spaces. Some basic attempts in this direction have been put forward in refs. [7, 9, 10].

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